S.-T. Yau College Student Mathematics Contests 2023

Oral Exams in Geometry and Topology

All-round (Solve 1 out of 2 problems)

1. Consider the two-dimensional unit sphere S^2. Let Δ denote the Laplace operator of 1-form on S^2. Find all smooth 1-forms ω that satisfy the equation

$$\Delta\omega + \omega = 0.$$

2. Let M^n be an $n(\geqslant 2)$-dimensional compact Riemannian manifold without boundary isometrically immersed in the Euclidean space $\mathbb{R}^{n+1}$. Then the first eigenvalue $\lambda_1(M^n)$ of the Laplacian on M^n satisfies

$$\lambda_1(M^n) \leqslant \frac{n}{\mathrm{Vol}(M^n)} \int_M H^2 d\mu.$$

Furthermore, the equality holds if and only if M^n is a round sphere in $\mathbb{R}^{n+1}$. Here $H = \frac{1}{n}\sum_{i=1}^n \kappa_i$ is the mean curvature of M^n in $\mathbb{R}^{n+1}$ and $\mathrm{Vol}(M^n)$ is the volume of M^n.